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# On the Hamiltonicity of Some Subgraphs in the Lattice of Binary Paths

Let $\mathcal{P}_n$, where $n$ is a positive integer, be the set of all binary paths $P$ of length $|P| = n$, i.e., lattice paths $P = p_1p_2\cdots p_n$ where each *step* $p_i$, $i \in [n]$, is either an *upstep* $u = (1,1)$ or a *downstep* $d = (1,-1)$ and connects two consecutive points of the path $P$. The number of $u$'s (resp. $d$'s) in $P$ is denoted by $|P|_u$ (resp. $|P|_d$). A maximal sequence of $u$'s (resp. $d$'s) in $P$ is called *ascent* (resp. *descent*) of $P$. The last point of an ascent (resp. descent) is called *peak* (resp. *valley*) of the path. Clearly, every peak (resp. valley) corresponds to either an occurrence of $ud$ (resp. $du$), or an occurrence of $u$ (resp. $d$) at the end of the path. It is convenient to consider that the starting point of a path is the origin of a pair of axes. The $y$-coordinate of a lattice point on $P$ is called *height* of this point.

A natural partial ordering on $\mathcal{P}_n$ is defined by the geometric representation of paths $P, Q \in \mathcal{P}_n$ where $P \leq Q$ whenever $P$ lies (weakly) below $Q$. We note that $Q$ covers $P$ whenever $Q$ is obtained from $P$ by turning exactly one of $P$'s valleys into a peak. It is well-known that the poset $(\mathcal{P}_n, \leq)$, or simply $\mathcal{P}_n$, is a finite, self-dual, distributive, graded lattice with minimum and maximum elements the paths $\mathbf{0}_n = d^n = \underbrace{dd\cdots d}_{n \text{ times}}$ and $\mathbf{1}_n = u^n = \underbrace{uu\cdots u}_{n \text{ times}}$ respectively. This lattice appears in the literature in various equivalent forms (e.g., sequences of integers [6], binary words [2, p. 92], subsets of $[n]$ [3], permutations of $[n]$ [8, p. 402], partitions of $n$ into distinct parts [7], threshold graphs [4]).

In this work, we consider the Hasse graph $G_n$ of $\mathcal{P}_n$, the edges of which are defined by the covering relation. The lattice $\mathcal{P}_n$ (resp. the graph $G_n$) is isomorphic to the lattice $M(n)$ (resp. the cover graph $A_n$ of $M(n)$), introduced by Stanley [6] (resp. considered by Savage et al. [5]). Furthermore, in [5] (working on the isomorphic graph $A_n$) it is proved that for every $n \geq 3$ the subgraph $G_n \setminus \{d^n, d^{n-1}u, u^n, u^{n-1}d\}$ is Hamiltonian. Clearly, this is the largest Hamiltonian subgraph of $G_n$, since the excluded vertices do not belong to any cycle. In a similar direction, Eades and Hickey [1] gave a sufficient and necessary condition for the subgraph of $G_n$ on the interval $[d^{n-k}u^k, u^k d^{n-k}]$ to have a Hamiltonian path (iff $k \leq 1$ or $k \geq n-1$ or $n$ is even and $k$ is odd).

In this work, it is shown that $G(P)$ is Hamiltonian for all paths $P$ that have at least two peaks and ending with an upstep, where $G(P)$ is the subgraph of $G_n$ induced by the interval $I(P) = [d^{n-2}ud, P]$, i.e., the interval which contains the elements of $\mathcal{P}_n$ less than or equal to $P$, excluding the first two elements of $\mathcal{P}_n$.

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