Michał Zwierzyński
Warsaw University of Technology

Parallel Hiking Trails in the Mountains on the Torus

Let's consider a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0) = f(1)$. Let's assume that f has finitely many local extrema and the values at these extrema are pairwise different non-integers modulo 1. Then, let $\hat{f}(x) = f(x) \bmod 1$ for $x \in [0,1]$. The graph of $\hat{f}$ we will call a *mountain*. Note that $\hat{f} : [0,1] \to [0,1]$ and if we glue the endpoints of the interval $[0,1]$, the graph of $\hat{f}$ is the subset of the torus. See Figure 1 for an example of a mountain.

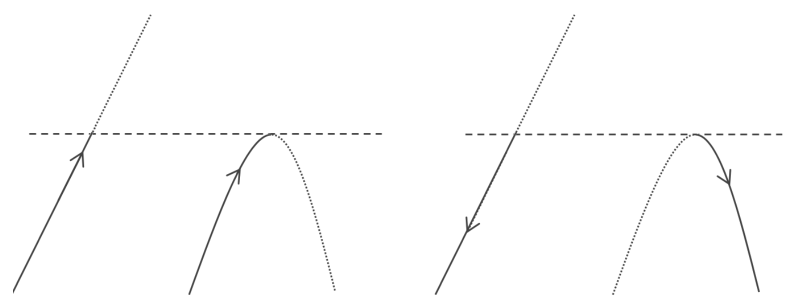

Figure 1: An example of a mountain, the hikers A and B, a peak s_0 and a valley s_1.

Now, let's imagine two points (the *hikers*) on a mountain m which is the graph of $\hat{f}$. The hikers can move on m but all the time they must be on the same level on m (like in Figure 1).

Now, we will add more restrictions for moving in the mountains. The hikers always move forward (left or right) except when one of them encounters a peak or valley (an extremum of $\hat{f}$) – then the other hiker turns back (see Figure 2). Thanks to this strange procedure, the hikers will be on the same level all the time.

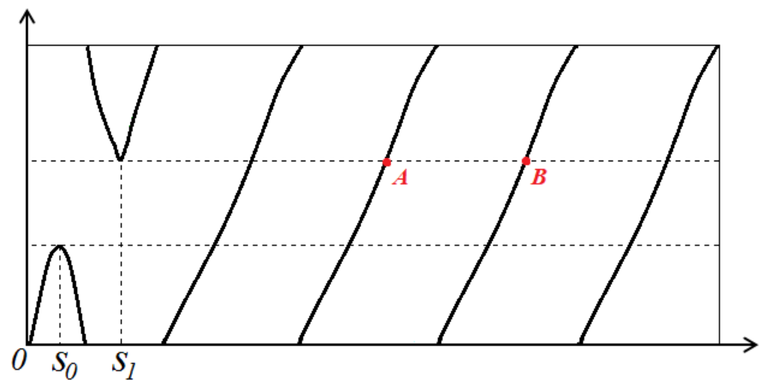

Figure 2: Turning back.

Finally, we are asking: how many different hiking trails are for a fixed mountain? If both hikers starts in the same peak of valley – will they ever meet again (if so, where?)? During the talk, we will get to know partial answers to the questions posed.

If we look at the list of articles below, we can get scared by the subject of differential geometry. This is because the problem described is related to the geometric properties of many sets such as the Wigner caustic, the Centre Symmetry Set, the secant caustic, the constant measure set. Any parallel hiking trail property found will result in some global geometry of said sets. However, during the presentation, we will only focus on mountain paths in isolation from differential geometry.

References

[1] W. Domitrz, M. Zwierzyński, *The Geometry of the Wigner Caustic and a Decomposition of a Curve into Parallel Arcs*, Analysis and Mathematical Physics 12, 2022, p.7.

[2] W. Domitrz, M. C. Romero Fuster, M. Zwierzyński, *The Geometry of the Secant Caustic of a Planar Curve*, Differential Geometry and its Applications 78, 2021, p.101797.

[3] M. Zwierzyński, *Isoperimetric Equalities for Rosettes*, International Journal of Mathematics 31(05), 2020, p.2050041.

[4] W. Domitrz, M. Zwierzyński, *Singular Points of the Wigner Caustic and Affine Equidistants of Planar Curves*, Bulletin of the Brazilian Mathematical Society, New Series 51, 2020, pp.11-26.

[5] W. Domitrz, M. Zwierzyński, *The Gauss-Bonnet Theorem for Coherent Tangent Bundles over Surfaces with Boundary and Its Applications*, Journal of Geometric Analysis 30, 2020, pp.3243–3274.